

# MEAN CURVATURE FLOW WITH FREE BOUNDARY - TYPE 2 SINGULARITIES

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**ABSTRACT.** In this paper we give sufficient conditions that guarantee the mean curvature flow with free boundary on an embedded rotationally symmetric double cone develops a Type 2 curvature singularity. We additionally prove that Type 0 singularities may only occur at infinity.

## 1. INTRODUCTION

We say that a smooth one-parameter family of immersed disks  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  evolves by the mean curvature flow with free boundary on a support hypersurface  $F_\Sigma : \Sigma \rightarrow \mathbb{R}^{n+1}$  if

$$(1) \quad \begin{aligned} \frac{\partial F}{\partial t} &= \vec{H} = -H\nu && \text{on } D^n \times [0, T) \\ \langle \nu, \nu_\Sigma \rangle &= 0 && \text{on } \partial D^n \times [0, T), \\ F(\partial D^n, t) &\subset F_\Sigma(\Sigma), \quad \text{and} \quad F(\cdot, 0) = F_0(\cdot). \end{aligned}$$

Local existence follows, as demonstrated by Stahl [11], by writing the evolving hypersurfaces as graphs for a short time over their initial data. Stahl additionally gave continuation criteria: a-priori bounds on the second fundamental form are sufficient for the global existence of a solution [13, 12]. In this work he also showed that initially convex data remains convex when the support hypersurface is umbilic, and that in this situation the flow contracts to a round hemispherical point (a Type 1 singularity). A generalisation to other contact angles of Stahl's continuation criteria was obtained by Freire [7].

Buckland studied a setting similar to that of Stahl, and focused on obtaining a classification of singularities according to topology and type [2]. Koeller has generalised the regularity theory developed by Ecker and Huisken [3, 4, 5] to the setting of free boundaries [8]. His main regularity theorem is a criterion under which the singular set will have measure zero.

The authors have studied initially graphical mean curvature flow with free boundary, obtaining long time existence results and results on the formation of curvature singularities on the free boundary [17, 15, 16]. A similar angle approach has been employed by Lambert [10] in his work. Edelen's work is the first systematic treatment of Type 2 singularities [6]. Convexity estimates play a fundamental role in his work. Edelen's work implies that rescaling a mean curvature flow with free boundary at a type 2 singularity yields a weakly convex mean curvature flow with free boundary in a hyperplane. After reflection a convex translating soliton is obtained which furthermore decomposes into a product of a strictly convex  $k$ -manifold and  $\mathbb{R}^{n-k}$ . This result implies that solutions satisfying the hypotheses of Theorem 1.2 also under rescaling have the same structure property. A family of examples exhibiting this behaviour is given in Remark 1, see also Figure 1.

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In previous work the second author has used results on the mean curvature flow of embedded discs inside generalised cylinders to answer questions of existence and uniqueness of minimal hypersurfaces [18]. There, non-existence is proved by establishing that the flow terminates in a curvature singularity developing at the apex of a pinching cylinder. A pinching cylinder is defined as follows.

**Definition 1.1.** Let  $\omega_\Sigma : Oz \rightarrow [0, \infty)$  be a continuous function. Assume that  $\omega_\Sigma$  is smooth outside finitely many points  $P = \{w_1, \dots, w_{n_p}\}$ , where  $\omega_\Sigma(w_i) = 0$ ; that is,  $\omega_\Sigma \in C_{loc}^\infty(Oz \setminus P)$ . Assume that there exists a compact set  $K \supset P$  such that

$$z \frac{d\omega_\Sigma}{dz}(z) > 0 \quad \text{for all } z \in Oz \setminus K.$$

The function  $\omega_\Sigma$  generates a smooth rotationally symmetric disconnected hypersurface  $F_\Sigma : \Sigma \rightarrow \mathbb{R}^{n+1}$ , where  $\Sigma$  is the disjoint union of  $n_p + 1$  cylinders. We term the support hypersurface  $F_\Sigma$  a pinching cylinder.

One result that guarantees the development of a finite-time singularity is:

**Theorem 1.2** (Flow in pinching cylinders [18]). *Let  $\Sigma$  be a pinching cylinder as in Definition 1.1 with  $n_p = 1$ . Let  $w_1 = z^* = 0$ . Assume that for all  $z \in Oz$*

$$(2) \quad \langle \nu_\Sigma(z), e_1 \rangle > C_\Sigma \geq 0$$

where  $C_\Sigma$  is a global constant and  $\nu_\Sigma$  is the normal to  $\omega_\Sigma$ . The graph condition (2) is understood as limits from above and below at points in  $P$ .

Suppose that for all  $z \in Oz \setminus \{0\}$ ,

$$(3) \quad z \frac{d\omega_\Sigma}{dz}(z) > 0.$$

Then the maximal time  $T$  of existence for any graphical mean curvature flow  $\omega : D(t) \times [0, T) \rightarrow \mathbb{R}$  with free boundary on  $\Sigma$  (see (5)) is finite. The hypersurfaces  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  generated by  $\omega$  contract as  $t \rightarrow T$  to the point  $(0, 0)$ .

In [18] sufficient conditions were given that guarantee the singularity will be Type 1 and Type 0. Type 0 singularities are a new notion introduced in that paper, where the flow becomes singular purely by the domain vanishing, with the second fundamental form remaining uniformly bounded.

This paper is concerned with an extension to further classification of the singularity and more importantly with the proof of existence of a Type 2 singularity for mean curvature flow with free boundaries.

**Theorem 1.3** (Type 2 singularities). *Let  $\omega_\Sigma$  and  $\omega_0$  be as in Theorem 1.2. If there exist three constants  $C_1, C_2, C_3 \in \mathbb{R}$  such that  $0 < C_1 < \infty$ ,  $C_1 < C_2 < \infty$ ,  $0 < C_3 < \infty$ , and two constants  $\alpha, \delta \in (0, \infty)$  satisfying  $\frac{2\delta}{\alpha+1} > 1$  such that for  $z$  sufficiently close to 0 we have:*

$$(4) \quad \frac{C_1}{z^\delta} \leq \left| \frac{\frac{d\omega_\Sigma}{dz}(z)}{\omega_\Sigma(z)} \right| \leq \frac{C_2}{z^\alpha}, \quad \text{and} \quad \left| \frac{d\omega_\Sigma}{dz}(z) \right| \leq C_3,$$

then the singularity from Theorem 1.2 is Type 2, in particular there exists  $C \in (0, \infty)$  such that for  $t$  sufficiently close to  $T$  we have

$$|A|^2(x, t) \geq \frac{C}{(T - t)^{\frac{2\delta}{\alpha+1}}}.$$

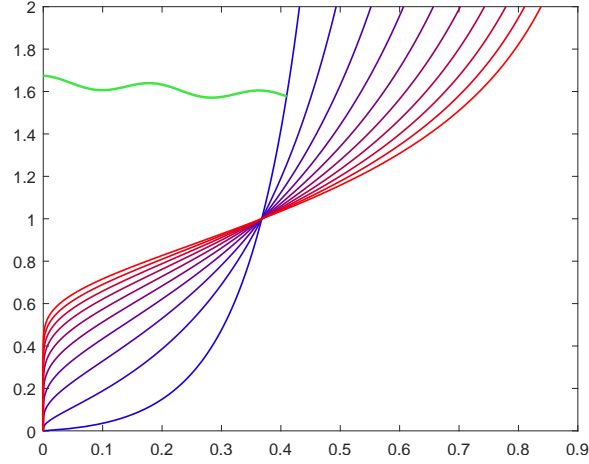


Figure 1: Profile curves for  $\omega_{k,\Sigma}$  for  $k = \frac{l}{4}, l = 1, \dots, 10$ . The colours move from blue to red as  $l$  increases. Sample initial data that moves to a type 2 singularity is in green.

*Remark 1* (Examples of  $\Sigma$  that produce a Type 2 singularity). Let us consider a family of pinching cylinders with profile

$$\omega_{\Sigma,k}(z) = \exp\left(-\frac{1}{z^k}\right), \quad k > 0.$$

Then

$$\left| \frac{\frac{d\omega_{\Sigma}(z)}{dz}}{\omega_{\Sigma}(z)} \right| = \frac{k}{z^{k+1}},$$

so (4) is satisfied with  $\alpha = \delta = k + 1$  and  $C_1 = C_2 = k$ . The second fundamental form along a graphical mean curvature flow with  $\omega_{\Sigma} = \omega_{k,\Sigma}$  blows up quickly, with the estimate

$$|A|^2(x, t) \geq \frac{C}{(T - t)^{\frac{2\delta}{\alpha+1}}} \geq \frac{C}{(T - t)^{1+\frac{k}{k+2}}},$$

for  $t$  sufficiently close to  $T$ . Interestingly, the rate of blowup for the second fundamental form is never as fast as  $(T - t)^{-2}$ , but can be made arbitrarily close. Figure 1 illustrates this example.

We finally show in the following theorem that Type 0 singularities may only occur at infinity, indicating that there may not be many more cases of Type 0 singularities than those already found in [18].

**Theorem 1.4** (Type 0 singularities). *Let  $\omega_{\Sigma}$  and  $\omega_0$  be as in Theorem 1.2. The singularity from Theorem 1.2 is not Type 0.*

This paper is organised as follows. Sections two and three contain definitions and prerequisites. In section four we discuss previous results for context and proceed with the proof of Theorem 1.3 and Theorem 1.4.

## 2. MEAN CURVATURE FLOW WITH FREE BOUNDARY SUPPORTED ON A GENERALISED CYLINDER

The behaviour of immersions flowing by the mean curvature flow with free boundary is largely unknown, with available results in the literature indicating that a complete picture of asymptotic

behaviour irrespective of initial condition is extremely difficult to obtain [13, 8]. Therefore the relevant question is: under which initial conditions is it possible to obtain a complete picture of asymptotic behaviour?

Working in the class of graphical hypersurfaces is a viable strategy, so long as the graph condition can be preserved [9, 14, 16, 15]. In each of these works, global results were enabled by symmetry of the initial data and/or of the boundary. Without such symmetries, recent work indicates that graphicality is not in general preserved [1] (even in the case where  $F_\Sigma(\Sigma)$  is a standard round sphere).

Let us formally set the support hypersurface  $F_\Sigma : \Sigma \rightarrow \mathbb{R}^{n+1}$  to be rotationally symmetric and generated by the graph of a function  $\omega_\Sigma : Oz \rightarrow \mathbb{R}$  over the  $Oz$  axis.

By convention we let  $x = (x_1, \dots, x_n)$  be a point in  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ , with  $n \geq 2$  and denote by  $y = |x|$  the length of  $x$ . With this convention the profile curve of the support surface lies in a plane generated by  $Oy$  and  $Oz$  axes. We write the graph condition on  $\omega_\Sigma$  as

$$\langle \nu_\Sigma(z), e_1 \rangle > C_\Sigma \geq 0,$$

where  $C_\Sigma$  is a global constant,  $\nu_\Sigma$  the normal to  $\omega_\Sigma$ , and  $\langle \cdot, \cdot \rangle$  is the standard inner product in  $\mathbb{R}^{n+1}$ . Our convention is that  $\nu_\Sigma$  points away from the interior of the evolving hypersurface.

Let us now describe how a rotationally symmetric graphical mean curvature flow with free boundary  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  satisfying (1) can be represented by the evolution of a scalar function (the graph function). Let us set  $D(t) = (0, r(t)) \subset \mathbb{R}$ . The Neumann boundary is at  $\partial D(t) = r(t)$ . The left-hand endpoint of  $D(t)$ , the zero, is not a true boundary point. It arises from the fact that the scalar generates a radially symmetric graph that is topologically a disk. The coordinate system degenerates at the origin and so it is artificially introduced as a boundary point. This is however a technicality, and no issues arise in dealing with quantities at this fake boundary point, since by symmetry and smoothness we have that the radially symmetric graph is horizontal at the origin.

We represent the mean curvature flow of a radially symmetric graph  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  by the evolution of its graph function  $\omega : D(t) \times [0, T) \rightarrow \mathbb{R}$  that must satisfy the following:

$$\begin{aligned} (5) \quad & \frac{\partial \omega}{\partial t} = \frac{d^2 \omega}{dy^2} \frac{1}{1 + (\frac{d\omega}{dy})^2} + \frac{d\omega}{dy} \frac{n-1}{y} && \text{on } (0, r(t)) \times [0, T), \\ & \langle \nu_\omega, \nu_\Sigma \rangle = 0 \text{ and } r(t) = \omega_\Sigma(\omega(r(t), t)) && \text{on } r(t) \times [0, T), \\ & \lim_{y \rightarrow 0} \frac{1}{y} \frac{d\omega}{dy}(y) \text{ exists, and} && \\ & \omega(y, 0) = \omega_0 && \text{on } (0, r(0)). \end{aligned}$$

Here  $\omega_0 : (0, r(0)) \rightarrow \mathbb{R}$  generates the initial graph  $\omega_0 \in C^2((0, r(0)))$  that also satisfies the Neumann boundary condition  $\langle \nu_{\omega_0}, \nu_\Sigma \rangle = 0$  at  $r(0)$ .

Note that in this representation the graph direction for  $\omega_\Sigma$  is perpendicular to the graph direction for  $\omega$ . (Contrast with [17].) The two graphs share the same axis of revolution. Examples of this include graphs evolving inside a vertical catenoid neck or inside the hole of a vertical unduloid.

### 3. EXISTENCE AND PREREQUISITES

Global existence of solutions to (5) under restrictions on  $\Sigma$  was shown in [18] by obtaining uniform  $C^1$  estimates. The problem (5) is a quasilinear second-order PDE on a time-dependent domain with a Neumann boundary condition. The change in domain can be calculated (see (7))

and depends only on  $\omega_\Sigma$ ,  $\omega'$ , and  $\omega''$ . The local unique existence of a solution in this setting is standard and has been discussed in detail in [14, 17].

The boundary condition  $\langle \nu_\omega, \nu_\Sigma \rangle = 0$  can be written in a simpler way if we take into account the fact that we are working with two graph functions. The outer normal to  $\omega$  is given by

$$\nu_\omega = \frac{1}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}} \left( -\frac{d\omega}{dy}, 1 \right).$$

For the unit normal to  $\omega_\Sigma$  we need to rotate and translate the axes. We find

$$\nu_\Sigma = \frac{1}{\sqrt{1 + \left(\frac{d\omega_\Sigma}{dz}\right)^2}} \left( 1, -\frac{d\omega_\Sigma}{dz} \right).$$

This transforms the Neumann boundary condition into

$$(6) \quad \frac{d\omega}{dy}(r(t), t) = -\frac{d\omega_\Sigma}{dz}(\omega(r(t), t)) \quad \text{for all } t \in [0, T),$$

and gives us the following uniform boundary gradient estimate for  $\omega$  by an upper bound on the gradient of  $\omega_\Sigma$ .

**Lemma 3.1** (Uniform boundary gradient estimates). *Let  $\omega_\Sigma$  and  $\omega_0$  be defined as above. Assume (2).*

*Then*

$$\left| \frac{d\omega}{dy}(r(t), t) \right| \leq \sqrt{\frac{1}{C_\Sigma} - 1}$$

for all  $t \in [0, T)$ .

*Remark 2.* On the free Neumann boundary, the rotational symmetry of the solution prevents tilt behaviour. This occurs when the normal to the graph becomes parallel to the vector field of rotation for  $\Sigma$ . This behaviour is explained in much greater detail in [14] and it is present in many situations of free boundary problems [1], thus the need to use the rotational symmetry in constructing the barriers needed to show the elliptic results.

The derivative of the boundary point  $r(t)$  is computed as follows. Since

$$r(t) = \omega_\Sigma(\omega(r(t), t)),$$

we calculate

$$r'(t) = \frac{d\omega_\Sigma}{dz} \left( \frac{\partial \omega}{\partial t} + \frac{d\omega}{dy} r'(t) \right).$$

Substituting in the boundary condition (6) and  $\frac{\partial \omega}{\partial t} = -Hv$  yields

$$(7) \quad r'(t) = -\frac{H}{v} \frac{d\omega_\Sigma}{dz},$$

where we have once again denoted  $v = \sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}$ , and at the boundary  $v = \sqrt{1 + \left(\frac{d\omega_\Sigma}{dz}\right)^2}$ .

The norm squared of the second fundamental form and mean curvature in terms of the profile curve  $\omega$  are expressed as in the following lemma.

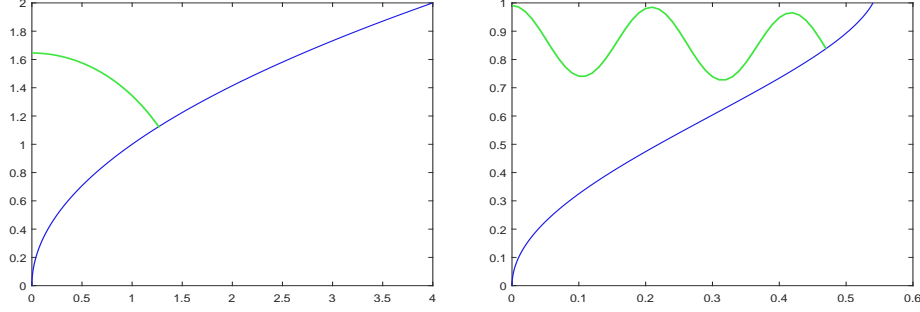


Figure 2: Examples of initial data that evolve toward finite-time singularities.

**Lemma 3.2.** *For a rotationally symmetric hypersurface generated by the rotation of a graph function  $\omega$  about an axis perpendicular to the graph direction, the norm squared of the second fundamental form and mean curvature are given by the formulae*

$$|A|^2 = \frac{1}{(1 + (\frac{d\omega}{dy})^2)^3} \left( \frac{d^2\omega}{dy^2} \right)^2 + \frac{1}{1 + (\frac{d\omega}{dy})^2} \frac{1}{y^2} \left( \frac{d\omega}{dy} \right)^2$$

$$H = -\frac{1}{\sqrt{1 + (\frac{d\omega}{dy})^2}^3} \frac{d^2\omega}{dy^2} - \frac{1}{\sqrt{1 + (\frac{d\omega}{dy})^2}} \frac{1}{y} \frac{d\omega}{dy}.$$

#### 4. TYPE 2 SINGULARITIES

We treat the case when the support hypersurface  $\Sigma$  pinches on its axis of rotation; that is, there exists one or more points  $z^*$  such that  $\omega_\Sigma(z^*) = 0$ . We do not require that  $\Sigma$  is smooth at those points so examples of such support hypersurfaces include cones, parabolae or hypersurfaces that form cusps at the rotation axis.

Let us recall Definition 1.1.

**Definition** (Pinching cylinder). *Let  $\omega_\Sigma : Oz \rightarrow [0, \infty)$  be a continuous function. Assume that  $\omega_\Sigma$  is smooth outside finitely many points  $P = \{w_1, \dots, w_{n_p}\}$ , where  $\omega_\Sigma(w_i) = 0$ ; that is,  $\omega_\Sigma \in C_{loc}^\infty(Oz \setminus P)$ . Assume that there exists a compact set  $K \supset P$  such that*

$$z \frac{d\omega_\Sigma}{dz}(z) > 0 \quad \text{for all } z \in Oz \setminus K.$$

*The function  $\omega_\Sigma$  generates a smooth rotationally symmetric disconnected hypersurface  $F_\Sigma : \Sigma \rightarrow \mathbb{R}^{n+1}$ , where  $\Sigma$  is the disjoint union of  $n_p + 1$  cylinders. We term the support hypersurface  $F_\Sigma$  a pinching cylinder.*

**Remark 3.** Although we require that  $\omega_\Sigma$  be only continuous on  $\mathbb{R}$ , it may pinch and be smooth (or analytic) everywhere on  $Oz$ . This is the case if  $\omega_\Sigma$  is a non-negative polynomial in  $z$  with zeros; for example,

$$\omega_\Sigma(z) = (z - 2)^2(z + 2)^2.$$

We also recall Theorem 1.2.

**Theorem** (Flow in pinching cylinders [18]). *Let  $\Sigma$  be a pinching cylinder as in Definition 1.1 with  $n_p = 1$ . Let  $w_1 = z^* = 0$ . Assume that for all  $z \in Oz$*

$$\langle \nu_\Sigma(z), e_1 \rangle > C_\Sigma \geq 0$$

*where  $C_\Sigma$  is a global constant and  $\nu_\Sigma$  is the normal to  $\omega_\Sigma$ . The graph condition (2) is understood as limits from above and below at points in  $P$ .*

*Suppose that for all  $z \in Oz \setminus \{0\}$ ,*

$$z \frac{d\omega_\Sigma}{dz}(z) > 0.$$

*Then the maximal time  $T$  of existence for any solution  $\omega : D(t) \times [0, T) \rightarrow \mathbb{R}$  to (5) is finite. The hypersurfaces  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  generated by  $\omega$  contract as  $t \rightarrow T$  to the point  $(0, 0)$ .*

*Remark 4* (Non-rotational initial data). Any initially bounded mean curvature flow with free boundary, irrespective of symmetry or topological properties, exists at most for finite time when supported on a pinching cylinder as in Theorem 1.2. This is because so long as the initial immersion is bounded, we may always construct a rotationally symmetric graphical solution such that the initial immersion lies between this solution and the pinchoff point  $(0, z^*)$ . The flow generated by this pair of initial data remain disjoint by the comparison principle, and as the rotationally symmetric solution contracts to a point in finite time, the flow of immersions must either develop a curvature singularity in finite time or contract to the same point (and possibly remain regular while doing so).

Singularities are classified as Type 0, 1 or 2. In [18], we classified most cases as being Type 1 or better, see Figure 3 for an illustration of the prototypical Type 1 singularity. Type 0 singularities are not curvature singularities at all but a loss of domain. The cases that allow us to do this are when the gradient of  $\omega_\Sigma$  is bounded. This includes cones and cusps.

**Definition 4.1** (Singularities). *Let  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  be a mean curvature flow with free boundary supported on a pinching cylinder. If there exists an  $\varepsilon > 0$  such that for all  $t \in (T - \varepsilon, T)$*

- *the second fundamental form is uniformly bounded, that is,*

$$|A|^2(x, t) \leq C < \infty,$$

*then we say the singularity is Type 0;*

- *the second fundamental form is uniformly controlled under parabolic rescaling, that is,*

$$|A|^2(x, t) \leq \frac{C}{T - t},$$

*then we say the singularity is Type 1;*

- *neither of the previous two cases apply, we say the singularity is Type 2.*

**Theorem 4.2** (Type 1 singularities). *Let  $\omega_\Sigma$  and  $\omega_0$  be as in Theorem 1.2. If there exist two constants  $0 < C_1 < \infty$  and  $C_2 < \infty$  such that for  $z$  sufficiently close to  $z^*$  we have:*

- *Conical pinchoff*

$$C_1 \leq \left| \frac{d\omega_\Sigma}{dz}(z^*) \right| \leq C_2,$$

*then the singularity from Theorem 1.2 is Type 1;*

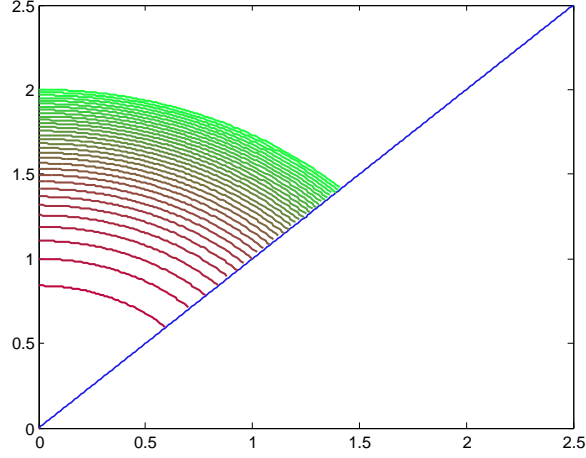


Figure 3: The mean curvature flow with free boundary on a standard cone with spherical initial data shrinks to a Type 1 singularity in finite time. The curves show a solution with initial radius 2, at times  $t = \frac{\tilde{t}}{(n-1)}$  where  $\tilde{t} \in \{0, 0.25, 0.50, 0.75, \dots, 8.0\}$ . Time moves forward as green becomes red. The solution begins moving slowly, however quickly speeds up. The amount of time to move from one leaf to the next is equal. The final dark red leaf shrinks to the origin in  $\frac{1}{4(n-1)}$  units of time.

- *Polynomial pinchoff*

$$C_1 |\omega_\Sigma(z)|^\sigma \leq \left| \frac{d\omega_\Sigma}{dz}(z) \right| \leq C_2 |\omega_\Sigma(z)|^\sigma,$$

for  $\sigma < 1$ , then the singularity from Theorem 1.2 is Type 1, and in particular there exist  $\hat{C}_1, \hat{C}_2$  such that for  $t$  sufficiently close to  $T$  we have

$$\frac{\hat{C}_1}{T-t} \leq |A|^2(x, t) \leq \frac{\hat{C}_2}{T-t}.$$

*Remark 5.* Conical pinchoff is a special case of polynomial pinchoff. For polynomial pinchoff, it isn't possible to satisfy all conditions of the theorem for  $\sigma \geq 1$ . For  $\sigma > 0$ , the pinchoff is *convex* and for  $\sigma < 0$  the pinchoff is *concave*. These names come from the following examples:

$$\omega_\Sigma(z) = z^\alpha$$

satisfies  $\omega'_\Sigma(z) = \alpha \omega_\Sigma^{1-\frac{1}{\alpha}}(z)$ . Therefore  $\alpha > 1$  corresponds to  $\sigma \in (0, 1)$  and  $\alpha < 1$  corresponds to  $\sigma < 0$ . Clearly all asymptotically polynomial pinchoffs are allowed by the condition  $\sigma < 1$ . Concave pinchoff is related to the singularity resulting from mean curvature flow with free boundary supported in the sphere, studied by Stahl [13].

**Theorem 4.3** (Type 0 singularities). *Let  $\omega_\Sigma : Oz \rightarrow \mathbb{R}$  be the profile curve of a rotationally symmetric hypersurface satisfying (2) and*

$$\lim_{z \rightarrow \infty} \omega_\Sigma(z) = 0, \quad \left| \frac{d\omega_\Sigma}{dz}(z) \right| \leq C |\omega_\Sigma|^{1+\sigma}(z), \quad \sigma > 0.$$



Then the maximal time of existence for any solution  $\omega : D(t) \times [0, T) \rightarrow \mathbb{R}$  to (5) satisfies  $T = \infty$ . The hypersurfaces  $F : D^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  generated by  $\omega$  satisfy

$$\|A\|_\infty^2(t) \rightarrow \alpha_0 \quad \text{as } t \rightarrow \infty,$$

and so either

- $F(D^n, t)$  converges smoothly to a flat disk; or
- Modulo translation,  $F(D^n, t)$  converges to a flat point, that is, a singularity of Type 0.

*Remark 6.* Examples of support hypersurfaces with profile curves satisfying the conditions of Theorem 4.3 include exponentials and reciprocal polynomials, such as

$$\omega_\Sigma(z) = e^{-z}$$

and a (monotone) mollification of

$$\omega_\Sigma(z) = \begin{cases} \frac{1}{z}, & \text{for } z > 1 \\ -z + 2, & \text{for } z \leq 1. \end{cases}$$

Theorem 1.3 brings Type 2 singularities into the picture. We now give its proof.

*Proof of Theorem 1.3.* As the gradient  $\omega_\Sigma$  is uniformly bounded, we obtain uniform bounds on the gradient of  $\omega$  as in [18]. Height bounds also follow exactly as in [18]. Since the mean curvature flow of graphs is a quasilinear second order parabolic PDE, this implies uniform bounds on higher derivatives, in particular there exists a  $C_4 \in (0, \infty)$  depending only on  $\omega_0$  and  $\omega_\Sigma$  such that

$$\left| \frac{d^2\omega}{dy^2} \right| \leq C_4.$$

Note that the singularity guaranteed by Theorem 1.2 occurs in spite of these estimates given by standard machinery of parabolic PDE as the domain shrinks away to nothing. As it does so, the curvature may explode at various rates, governed primarily by the blowup rate of  $\frac{1}{\omega}$ , a quantity that the standard theory does nothing to control.

This also implies that the graphs are smooth and exist up to the time of shrinking to the pinching point, remaining graphs. The maximum principle implies that the second fundamental form is bounded everywhere by its values on the boundary, see [4]. We estimate

$$\begin{aligned} \sup_{y \in [0, r(t)]} |A|^2(y, t) &= |A|^2(r(t), t) \\ &= \frac{1}{(1 + (\frac{d\omega}{dy})^2)^3} \left( \frac{d^2\omega}{dy^2} \right)^2(r(t), t) + \frac{1}{1 + (\frac{d\omega}{dy})^2} \frac{1}{r(t)^2} \left( \frac{d\omega}{dy} \right)^2(r(t), t) \\ &\geq C_\Sigma \frac{1}{r(t)^2} \left( \frac{d\omega}{dy} \right)^2(r(t), t) \\ &= C_\Sigma \frac{1}{\omega_\Sigma(z)^2} \left( \frac{d\omega_\Sigma}{dz} \right)^2(z) \\ &\geq C_\Sigma C_1 \frac{1}{z^{2\delta}} \\ &= C_\Sigma C_1 \frac{1}{\omega^{2\delta}(r(t), t)}, \end{aligned} \tag{8}$$

where we have used the lower bound on the ratio of  $\omega_\Sigma$  to its gradient for all  $z$  sufficiently close to 0 and Lemma 3.1 to estimate

$$\frac{1}{1 + \left(\frac{d\omega}{dy}\right)^2} \geq \frac{1}{1 + (1/C_\Sigma - 1)} = C_\Sigma.$$

We note that  $z = \omega(r(t), t)$  so we determine the speed at which  $\omega$  decreases on the boundary to find out more information on the asymptotic behaviour of  $|A|^2$ .

We use the parabolic evolution for  $\omega$  and the time evolution for  $r(t)$  from (7) to compute

$$\frac{d\omega}{dt} = \partial_t \omega + \frac{d\omega}{dy} r'(t) = -\frac{H}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}}.$$

Substituting the formula for the mean curvature in Lemma 3.2, we obtain

$$\frac{d\omega}{dt} = \frac{1}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}^4} \frac{d^2\omega}{dy^2} + \frac{1}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}^2} \frac{1}{y} \frac{d\omega}{dy}.$$

Using the Neumann boundary condition (6) and the upper bound on the quotient of  $\omega_\Sigma$  and its derivative we obtain

$$\frac{d\omega}{dt} = \frac{1}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}^4} \frac{d^2\omega}{dy^2} - \frac{1}{\sqrt{1 + \left(\frac{d\omega}{dy}\right)^2}^2} \frac{1}{\omega_\Sigma} \frac{d\omega_\Sigma}{dz} \geq -C_4 - \frac{C_2}{z^\alpha}.$$

We have also used that the second derivative is bounded by  $C_4$  and thus there exists a  $t^*$  such that for  $t^* < t < T$  we may multiply by  $z^\alpha$  to absorb the first term into the second above:

$$\frac{d\omega}{dt} \omega^\alpha \geq -C_4 \omega^\alpha - C_2 \geq -C_5,$$

for some positive  $C_5 > 0$ .

Integrating from  $t < T$  to  $T$  and using the fact that  $\omega(T) = 0$  we find

$$\frac{1}{\omega^{\alpha+1}} \geq \frac{1}{C_5(\alpha+1)} \frac{1}{T-t},$$

for all  $t \geq t^*$ . Substituting this into (8) we obtain the following bound for the second fundamental form

$$\sup_{y \in [0, r(t)]} |A|^2(y, t) \geq C \frac{1}{(T-t)^{\frac{2\delta}{\alpha+1}}},$$

for all  $t \in (t^*, T)$  and  $C = C_\Sigma C_1 (C_5(\alpha+1))^{-\frac{2\delta}{\alpha+1}}$ , that is, the singularity is Type 2, given that  $\frac{2\delta}{\alpha+1} > 1$ .  $\square$

We complete the paper by giving the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Let us assume that the singularity produced by Theorem 1.2 is type 0. Then there exists a  $\tilde{C} < \infty$  such that for all  $t \leq T$  we have

$$|A|^2(x, t) \leq \tilde{C}.$$

Now by Lemma 3.2 this implies

$$\frac{1}{(1 + \left(\frac{d\omega}{dy}\right)^2)^3} \left(\frac{d^2\omega}{dy^2}\right)^2 + \frac{1}{1 + \left(\frac{d\omega}{dy}\right)^2} \frac{1}{y^2} \left(\frac{d\omega}{dy}\right)^2 \leq \tilde{C}.$$

Now as noted in the proof of Theorem 1.3 the gradient of  $\omega$  is uniformly bounded and so there exists a constant  $0 < C < \infty$  such that for  $z$  sufficiently close to 0 we have:

$$\left| \frac{\frac{d\omega_\Sigma}{dz}(z)}{\omega_\Sigma(z)} \right| \leq C,$$

and  $\omega_\Sigma(0) = 0$ . Furthermore, by (3) there exists an  $\varepsilon > 0$  such that for all  $z \in (0, \varepsilon)$

$$\frac{d\omega_\Sigma}{dz}(z) > 0.$$

Therefore we have

$$\frac{d\omega_\Sigma}{dz}(z) \leq C|\omega_\Sigma(z)| = C\omega_\Sigma(z), \quad z \in (0, \varepsilon).$$

The second equality follows by smoothness of the generated hypersurface  $\Sigma$ ; if  $\omega_\Sigma$  crossed the axis of revolution then  $\Sigma$  would be singular.

Now the differential form of Grönwall's inequality applies to give

$$\omega_\Sigma(z) \leq \omega_\Sigma(0)e^{\int_0^z \tilde{C} dw} = 0.$$

Therefore  $\omega_\Sigma(z) = 0$  for all  $z \in [0, \varepsilon/2]$ , which is a contradiction with a variety of assumptions, for example  $n_p = 1$  and (3).  $\square$

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